Waging Simple Wars: A Complete Characterization of Two Battlefield Blotto Equilibria

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Abstract

In situations such as political and military campaigns, two players must allocate resources across multiple contests. We analyze these environments in terms of the canonical Colonel Blotto game, where two officers simultaneously allocate their forces across multiple fronts, attempting to win battles. Prior work on Blotto games has generally been quite technical, and often only provided example equilibria. We remedy both issues simultaneously by providing an intuitive, graphical algorithm for constructing the *complete* set of equilibria to all two battlefield Blotto games. We show how our method easily extends to address previously unsolved games. We find the complete set of equilibria to one generalization of the game and a large class of equilibria when relaxing the constant-sum assumption. We then discuss several other ways our results could aid further Blotto research.

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1. Introduction

In Gross and Wagner's (1950) canonical paper, two officers, Colonel Blotto and Enemy, are each endowed with a quantity of B and E soldiers, respectively. They compete on multiple battlefields, simultaneously deciding how to allocate

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[2] The ideas and views expressed herein do not necessarily reflect those of the United States Air Force or any other governmental organization.

their soldiers across each. The officer with more resources on a particular front wins that battle, and an officer's payoff is the sum of the values of the fronts won.[3]

These Colonel Blotto games have a wide range of potential applications including: military and political campaigns, network defense, and strategic hiring situations such as pro sports or the economics job market. For example, consider two Economics departments that have received "use-it or lose-it" grants to hire new faculty and suppose they are both interested in hiring the same candidates. Should the departments make many modest offers or fewer offers with higher salaries? To which candidates should they make which offers?

When studying Colonel Blotto interactions, researchers would like to know what strategies to expect in equilibrium. Perhaps just as importantly, we would like to know what types of behavior should never occur. In that vein, we completely characterize the set of Nash Equilibria in all two battlefield versions of this game.

Our characterization comes from a simple graphical algorithm which we think provides important intuition to this long studied game. Despite nearly a century of research, the literature's understanding of Blotto games is still limited. Perhaps much of this deficit can be attributed to the dimensionality of the problem. A strategy for either player is a *joint* distribution over each of the battlefields, and a Nash Equilibrium is then a pair of joint distributions.

Borel (1921) posited the first Colonel Blotto game nearly 90 years ago, but the first large set of solutions came from Gross and Wagner (1950). They provide example equilibria to many versions of the game. Much of the subsequent research explored the implications of modifying the standard Blotto game. For instance, Blackett (1954, 1958); Golman and Page (2009) all examine Blotto games where the payoff on an individual battlefield changes continuously as players alter their allocations, instead of discretely as one player outspends the other. Work by Kovenock and Roberson (2007), Powell (2009) and Roberson and Kvasov (2008) consider other modifications to the game, including relaxing the constant-sum assumption. Szentes and Rosenthal (2003) provide a bridge to the auction literature and compare solutions to some Blotto games to solutions to their "chopstick" auctions.[4]

While a few example equilibria to canonical Blotto games were provided over the 56 years[5] following Gross and Wagner's (1950) work, the next major theoretical contribution to the classic game came from Roberson (2006) who examines Blotto games on three or more *equally valued* battlefields. In most cases, he provides a complete characterization of the univariate marginal distributions of each players' strategies in any Nash Equilibrium.[6] Additionally, he

[3]We use the terms "battlefield" or "front" to refer to the location where the individual "battles" occur.

[4]Three chopsticks are being auctioned, and winning only one is worthless.

[5]e.g. see Weinstein (2005).

[6]He also uses these marginals to construct joint distributions that are a part of some equilibrium. This method does leave room for other, uncharacterized, joint distributions

finds example equilibria in the other cases.[7]

We return to the two battlefield canonical game and introduce an intuitive algorithm which characterizes the complete set of equilibria for the two battlefield game for any relative level of resources. Importantly, our paper deals with Blotto games where thus far we've at best had only example equilibria. In fact, it turns out that our algorithm deals quite easily with versions of the game where previously no equilibria were known.

Roberson (2006) examines Blotto play on more than two battlefields, but does not consider possible asymmetries across the fronts. We make an opposite trade-off and examine Blotto games with asymmetric battlefields, but restrict our analysis to two fronts.[8]

Our work makes two main contributions to the literature. First, we provide a simple graphical method for finding the complete set of equilibria to any two battlefield Blotto game. Our method provides important intuition previously missing from Blotto research. In fact, we start with the simple logic of the trivial case and extend it to cases that otherwise seem quite complicate. Second, we find the complete set of Nash Equilibria under a previously unsolved generalization of the Blotto game,[9] as well as a characterize a large class of equilibrium under a generalization relaxing the constant-sum assumptions.

The organization for the remainder of this paper is as follows: Section 2 formally describes the game we will start with. Section 3 provides the graphical algorithm we use to construct the complete set of equilibrium in the case of equally weighted battlefields. Section 4 demonstrates how to extend this algorithm to construct a similar set of equilibrium in two previously unsolved games: non-constant-sum Blotto games, and games where one player's forces may be relatively more effective on a particular front. Section 5 formalizes our intuition. Section 6 discusses other ways our results can aid further Blotto research and Section 7 Concludes.

2. Model

We start by analyzing the following version of Gross and Wagner's (1950) game. The two players, Blotto and Enemy, have a continuously divisible quantity of soldiers, B and E, available, respectively. Without loss of generality, normalize $B = 1$ and assume that Blotto is the advantaged player ($E < 1$).[10] Define Blotto's advantage as $\delta \equiv 1 - E$. In a two battlefield game players simultaneously set, $\{b_i\}_{i=1}^{2}$ and $\{e_i\}_{i=1}^{2}$, denoting Blotto and Enemy's respective

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which may also be a part of some equilibrium, but they must yield his marginals.

[7]Interestingly, extending our analysis to 3 or more battlefield games would seem to apply most directly to the case where he only provides example equilibria. For more see Section 6

[8]Thomas (2009) takes a third approach to this trade-off. She allows for asymmetric fronts and more than two battlefields, but she assumes that $B = E$.

[9]Interestingly, we completely characterize the set of equilibria in terms of *joint* strategy distributions, not just the corresponding *marginal* distributions. To our knowledge, this has not yet been done for any non-trivial canonical Blotto games.

[10]$E = 1$ is trivial.

Table 1: Objectives and Constraints

	Blotto	Enemy
Objective	$\max_{b_1,b_2}\left(\sum_{i=1}^{2}(Prob(b_i \geq e_i \mid \mu_E))\right)$	$\max_{e_1,e_2}\left(\sum_{i=1}^{2}(Prob(e_i > b_i \mid \mu_B))\right)$
Constraints	$b_1 \geq 0$, $b_2 \geq 0$ and $b_1 + b_2 \leq 1$	$e_1 \geq 0$, $e_2 \geq 0$, and $e_1 + e_2 \leq E$

Table 2: Payoffs by Region

Region	Bounds	Blotto's Payoff	Enemy's Payoff
i	$E \in [0, \frac{1}{2}B]$	2	0
ii	$E \in (\frac{1}{2}B, \frac{2}{3}B]$	$\frac{3}{2}$	$\frac{1}{2}$
iii	$E \in (\frac{2}{3}B, \frac{3}{4}B]$	$\frac{4}{3}$	$\frac{2}{3}$
iv	$E \in (\frac{3}{4}B, \frac{4}{5}B]$	$\frac{5}{4}$	$\frac{3}{4}$
n	$E \in (\frac{n-1}{n}B, \frac{n}{n+1}B]$	$\frac{n+1}{n}$	$\frac{n-1}{n}$

allocations to Battlefield i, subject to their resource constraints. The player with more soldiers on Battlefield i wins that battle and receives payoff 1. The losing player receives a payoff of 0 on that front. A strategy for Blotto, or Enemy is then a randomization over pairs of b_1 and b_2, or e_1 and e_2 respectively. Blotto's allocation are constrained by $b_1 \geq 0$, $b_2 \geq 0$ and $b_1 + b_2 \leq 1$. Enemy is constrained by $e_1 \geq 0$, $e_2 \geq 0$, and $e_1 + e_2 \leq E$.

A strategy for Blotto or Enemy can be represented with a probability measure μ_B or μ_E, respectively.[11] Table 1 provides the objective functions and resource constraints for this game. You can see from the payoff functions that, in the case of a tie we assume the battlefield goes to Blotto. This assumption is standard in most of the literature (Kvasov, 2007).[12]

Given this formal construction, equilibrium payoffs of the game are already known for any E. Figure 1 displays the possible resource endowments for Blotto and Enemy and shows the separate payoff regions. (It also relaxes the normalization that $B = 1$.) The payoffs and bounds for each region are displayed in Table 2 as an aid to the reader.[13]

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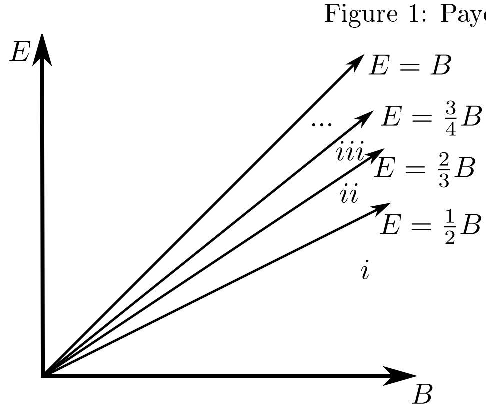
[11]For readers not familiar with probability measures, just think of μ_B as a function which takes a region in (b_1, b_2) space and returns the probability that a point in the region will be played, given Blotto's randomization. For example, if half the time Blotto plays allocations in S, then $\mu_B(S) = \frac{1}{2}$.

[12]This assumption is not important. If we instead assumed ties were decided by the flip of a fair coin, we could redo our analysis just by swapping some weak and strict inequalities.

[13]These payoffs and bounds were first reported in Gross and Wagner (1950). However, a reader familiar with their work may notice two differences. We have swapped the weak and strict inequalities in our bounds when compared to their results. This is due to our different assumptions regarding ties. We are also using different normalizations of the objective functions that we find more intuitive. So, we've normalized the payoffs appropriately.

Figure 1: Payoff Regions



3. Equilibrium Construction Intuition

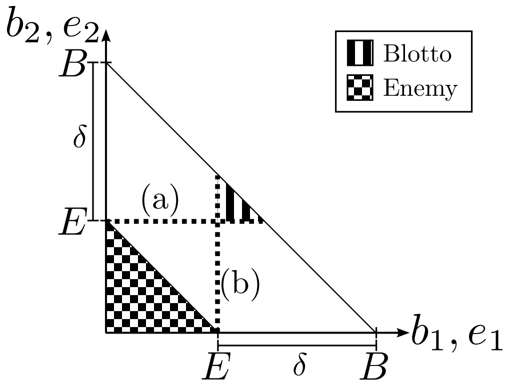
In this section we provide the intuition behind our algorithm. We leave the formal characterization for Section 5. Section 3.1 graphically constructs the set of all Nash equilibrium in the trivial case where $E \leq \frac{1}{2}B$ (Region 1). Section 3.2 constructs the slightly more complicated set of all Nash equilibrium in the case where $\frac{1}{2}B < E \leq \frac{2}{3}B$ (Region 2). Section 3.3 shows how we can continue using our method in other regions.

3.1. The Trivial Region (Region 1)

We begin considering the case where Blotto has at least double the resources of Enemy, $E \leq \frac{1}{2}B$. The results here may seem trivial, and in fact they are. However, the method of analysis we use here proves useful when considering more complicated versions of the game. The intuition for the region is simple. If Colonel Blotto has twice the forces of Enemy, he can guarantee himself victory on both battlefields by deploying forces (E, E) and earning payoff of 2. In fact, any allocation where Blotto allocates at least E to both battlefields will ensure Blotto a payoff of 2.

Figure 2 shows all feasible Blotto and Enemy allocations on one graph. On the graph, Blotto could potentially play any allocation within the simplex bounded by the axes and his full expenditure boundary (the line from $(0, B)$ to $(B, 0)$, representing his budget constraint). Similarly, Enemy could potentially play any allocation within the simplex bounded by the axes and his full expenditure boundary (the line from $(0, E)$ to $(E, 0)$). A strategy for either player is a randomization over his feasible allocations. Therefore, we take the general approach of finding the set of points over which each player may randomize in some equilibrium. It turns out that these sets are easily graphed areas. Subject to their constraints, Blotto can play any strategy that always sends at least E to both battlefields (the striped region in Figure 2), and Enemy can play any strategy at all (the checkered region). When Blotto plays in this manner, Enemy can never win on either battlefield.

Figure 2: The Trivial Region



As already mentioned, Blotto should ensure victory on both fronts. In order to see where Blotto can do so, we bound the area labeled Blotto in Figure 2 with three lines. Playing above the horizontal dotted line (a) ensures that Blotto plays at least E on Battlefield 2 and guarantees victory there. Playing to the right of the vertical dotted line (b) ensures that Blotto plays at least E on Battlefield 1, guaranteeing victory there. Finally the line from $(1, 0)$ to $(0, 1)$ ensures that Blotto plays only feasible allocations. When playing inside this region Blotto guarantees himself victory on both battlefields while respecting his resource constraint. When playing outside this region Blotto is either presenting Enemy with an opportunity to take one of the battlefields, or violating his resource constraint. Therefore, the complete set of equilibrium Blotto strategies is the set of strategies where Blotto randomizes over this region and this region only. As you can the only restriction on Enemy's strategy is that he must play within his constraints.

3.2. Region 2

Region 2 is somewhat more complicated and consists of all possible resource endowments where Enemy has more than half of Blotto's resources ($E > \frac{B}{2}$), but no more than two-thirds ($E \leq \frac{2B}{3}$). In region 1 Blotto was able to guarantee victory on both fronts by sending at least E soldiers to each battlefield.

While Blotto no longer has enough soldiers to guarantee victory on both fronts, he has enough resources to do the following: choose a battlefield by the flip of a fair coin; send E soldiers to that front and $\frac{E}{2}$ to the other front. The best that Enemy can do against this strategy is send more than half of his forces (for example all his forces) to one front. Enemy hopes he chooses the front to which Blotto sent $\frac{E}{2}$ in which case he will win one battle. Otherwise, he will find the bulk of his forces facing E of Blotto's soldiers, and less than half his troops facing $\frac{E}{2}$ of Blotto's. In this case he loses both battles.

Intuitively, in equilibrium, each player heavily attacks one front and sends a smaller force to the other front. Blotto hopes they both heavily attack the same front, while Enemy hopes they mismatch. Half the time Blotto wins both, and the rest of the time they each win one.

There are many strategies which loosely fit our above description. In order to construct the complete set of equilibrium strategies, we now consider this region graphically in Figure 3. Figure 3a shows that Blotto has two separate areas where he may play while attacking one battlefield heavily. Line (a) shows that he must play more than E on Battlefield 2 if he is to ensure victory there when he attacks that front heavily. Line (b) shows a similar condition when he decides to attack the other front heavily. If Blotto plays in each of these areas with a 50/50 chance, Enemy will not know which battlefield to avoid.

Figure 4b shows that when Enemy attacks a front heavily, he needs to ensure that he will win there when Blotto attacks the other front heavily. For instance, the horizontal line (c) shows that when Enemy is attacking Battlefield 2 heavily, he needs to be sure his force is large enough to beat any force Blotto might send to Battlefield 2 when Blotto is attacking Battlefield 1 heavily.[14] The vertical line (d) shows a similar condition when Enemy decides to attack Battlefield 1 heavily. If Enemy plays in each of the two regions with equal probability Blotto wont know which battlefield to attack heavily, and which only needs a smaller force.

When Blotto attacks a battlefield heavily, he needs to ensure he has enough left over troops to ensure victory on the other front when Enemy sends his smaller force there. For instance, randomizing between playing $(B, 0)$ and $(0, B)$ would not be a good idea. Enemy could play $(\frac{E}{2}, \frac{E}{2})$ and always win one battle. Figure 3c demonstrates this restriction. The horizontal line (e) shows that when Blotto attacks Battlefield 1 heavily, he needs to send enough forces to Battlefield 2 to ensure victory there if Enemy attacks Battlefield 1 heavily, but still sends a small force to Battlefield 2. The vertical line (f) demonstrates a similar condition when Blotto attacks Battlefield 2 heavily.

We now have two areas for both Blotto and Enemy, and we know they must play in both areas $\frac{1}{2}$ of the time. We now need to know how they may choose to distribute mass within those two areas. In equilibrium Blotto expects a payoff of $1\frac{1}{2}$ and Enemy expects $\frac{1}{2}$. The only condition on the distribution of mass within the two areas is that it cannot provide the other player with a higher expected payoff if they deviate. We demonstrate these conditions in Figure 4. Figure 4a demonstrates how certain Blotto distributions could provide Enemy with profitable deviation from his prescribed strategy. Observe the potential deviating (e_1^*, e_2^*). Line (h) divides the area where Blotto is attacking Battlefield 2 heavily into two parts. In j, $b_1 < e_1^*$, and in m, $b_1 \geq e_1^*$. Line (g) similarly divides the area where Blotto is attacking Battlefield 1 heavily. In k, $b_2 < e_2^*$ and in l, $b_2 \geq e_2^*$. We know that Enemy can only expect a payoff of at most $\frac{1}{2}$ if he were to play here.

Playing (e_1^*, e_2^*) he may end up winning either front. He may win on Bat-

[14] Recall that Enemy can never when on Battlefield 2 when Blotto is also attacking Battlefield 2 heavily.

Figure 3: Region 2 Construction

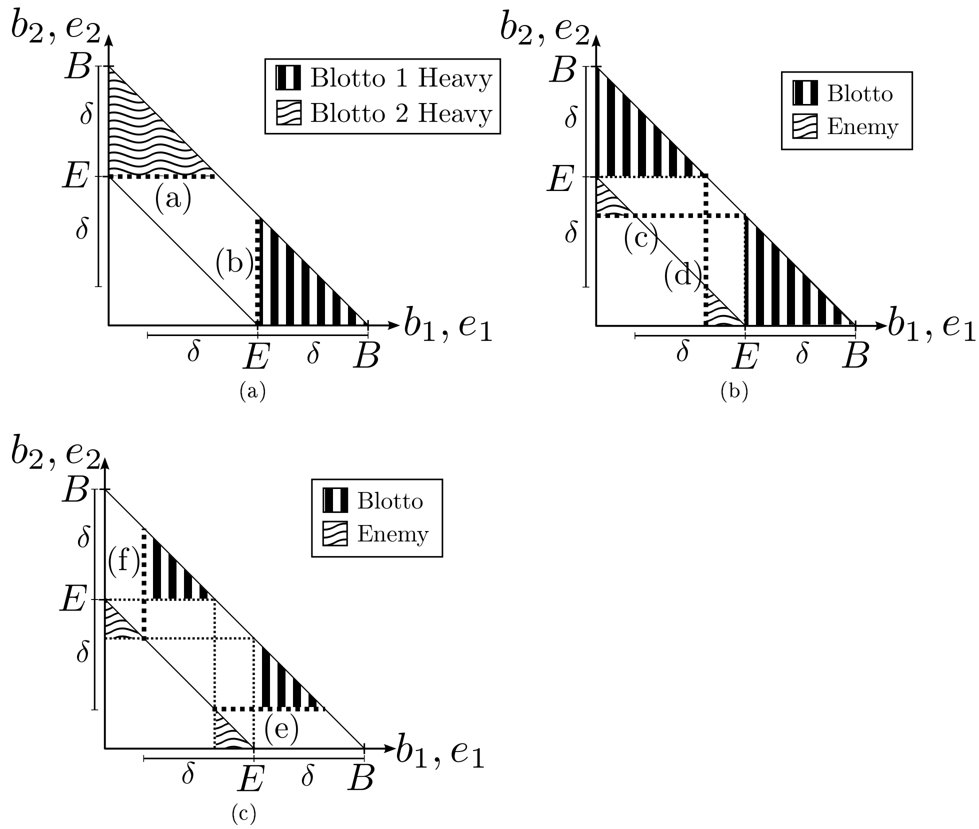
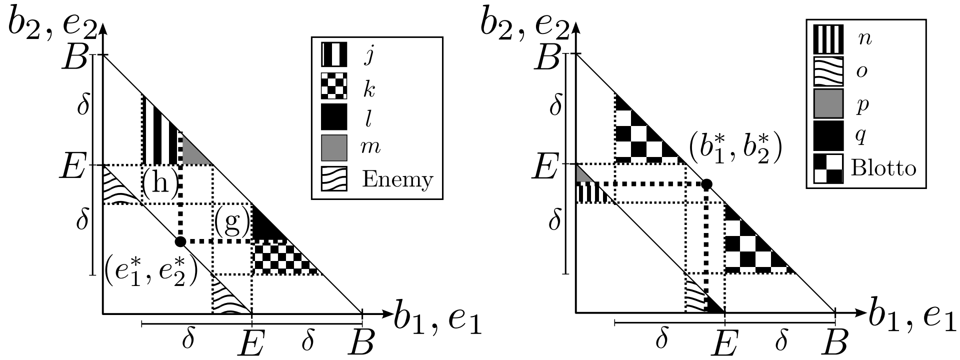


Figure 4: Region 2 Mass Restrictions



(a) (b)

tlefield 1 when Blotto plays in region j, and he may win on Battlefield 2 when Blotto plays in region k. In either of these cases he will win one battlefield. When Blotto plays in either of the other regions (l or m) Enemy loses on both fronts. Therefore, the total mass Blotto plays over regions j and k can be no more than $\frac{1}{2}$. Figure 4b demonstrates how similar restrictions affect Enemy's potential distributions.

Notice that we chose (e_1^*, e_2^*) (and (b_1^*, b_2^*)) somewhat arbitrarily. Any full expenditure deviating (e_1^*, e_2^*) would have done. As such, there are a continuum of such restrictions on how Blotto and Enemy can randomize over the two areas. However, we now have graphical representations of the complete set of equilibrium Blotto and Enemy strategies. They are the set of randomizations that place $\frac{1}{2}$ of the mass on the player's two areas shown above, and distribute the mass within those two areas in such a way that their opponent has no profitable deviations (as shown in Figure 4).

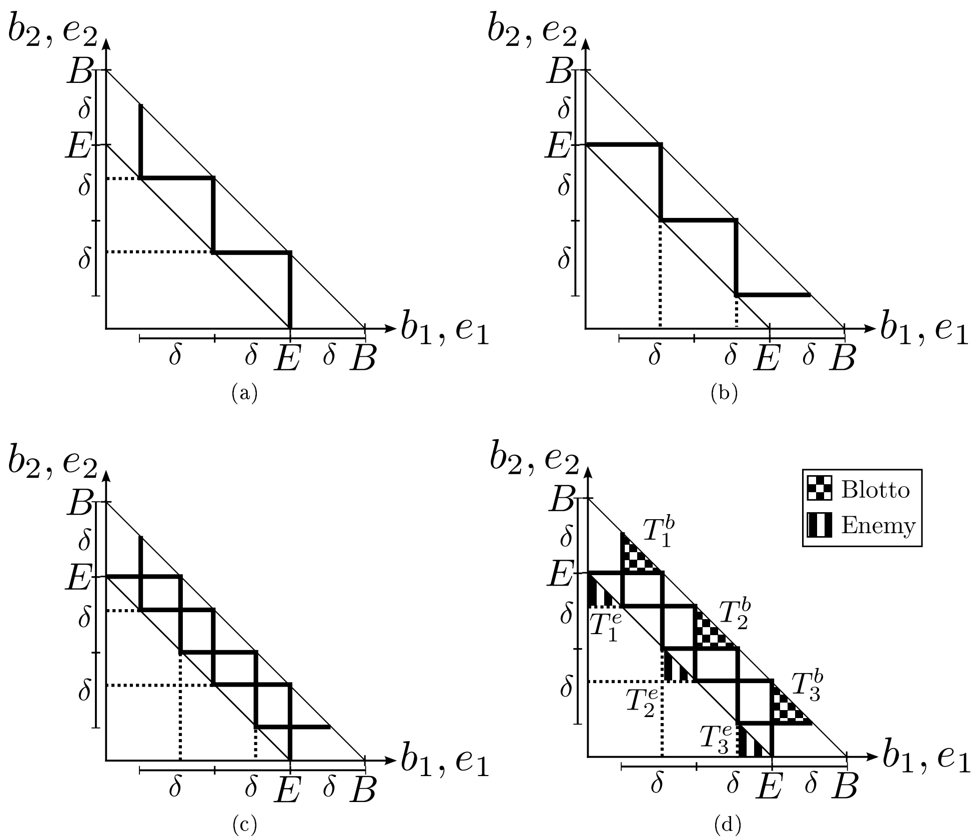
3.3. The General Approach

Here we will explain our graphical algorithm for finding the set of equilibrium in a generic region n. Recall that in region n we have $E \in (\frac{n-1}{n}B, \frac{n}{n+1}B]$. As an aid to the reader, we provide an illustration of our method for $n = 3$ in Figure 5.

Our graphical algorithm for finding the set of equilibrium to any two battlefield Blotto game is a three step process. In step one, as shown in Figure 5a, we draw a (solid) vertical line coming out of the point $(E, 0)$. Every time this line intersects a resource constraint we reflect it 90 degrees. We can stop the reflections once we reach a point (x_1, x_2) on Blotto's resource constraint where $x_2 > E$. Every time the solid line intersects Enemy's resource constraint we also draw the line's continuation (shown as a dotted line).

Step two is similar to Step one. As in Figure 5b, we draw a (solid) horizontal line coming out of the point $(0, E)$. Again, when the line intersects a resource

Figure 5: The General Graphical Method



constraint we reflect it 90 degrees. We can now stop the reflections once we reach a point (x_1, x_2) on Blotto's resource constraint where $x_1 > E$. We again draw a dotted line showing how the solid line would continue every time it intersects Enemy's resource constraint. After doing both steps one and two we will have a graph like Figure 5c.

In step three we use our graph to find the regions over which each player will randomize. After completing steps one and two we have n triangles directly below Enemy's resource constraint and $2n-1$ triangles directly below Blotto's resource constraint as shown in Figure 5c.[15] We label the triangles directly below Enemy's resource constraint $T_1^e, ..., T_n^e$, from top left to bottom right as in Figure 5d. These will be the areas over which Enemy can randomize in equilibrium. For the triangles directly below Blotto's resource constraint we label the top left T_1^b. As we move down and right along the resource constraint we skip the next triangle and then label T_2^b, skip another, label T_3^b and so on until we reach T_n^b. We can see this in Figure 5d. In equilibrium each player will play in each of their T_i's with probability $\frac{1}{n}$.[16]

If they play this way, with probability $\frac{1}{n}$ they will "match," or for some j Blotto and Enemy will play allocations in T_j^b and T_j^e, respectively. In this case, Blotto wins both battlefields. Otherwise (with probability $\frac{n-1}{n}$) they will "mismatch." Blotto and Enemy will play allocations in T_i^b and T_j^e, respectively, such that $i \neq j$. In this case each player wins one front. Thus, for any allocation in one of their T_i's, Blotto expects a payoff of $\frac{n+1}{n}$ while Enemy expects $\frac{n-1}{n}$.

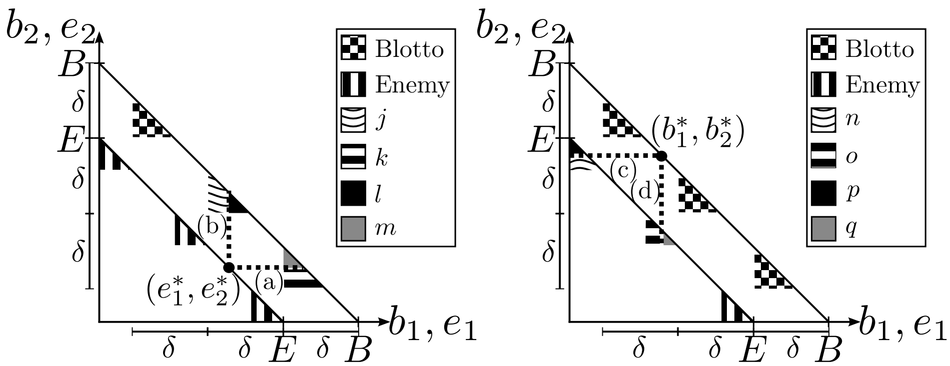
Before we can say we've completely characterized the set of equilibrium strategies, we need to discuss additional restrictions on how player's can randomize within their T_i's. These restrictions are very similar to the restrictions shown earlier in Figure 4; they prevent possible deviations by the opponent. Since either player's expected payoff is weakly increasing in allocations on either battlefield, ensuring that there are no full expenditure, expected payoff increasing deviations is sufficient.

Consider, $e^* = (e_1^*, e_2^*)$, a generic full expenditure deviation by Enemy like that shown in Figure 6a. Any such e^* will always lie on Enemy's resource constraint between some T_i^e and T_{i+1}^e. To understand how this deviation will affect Enemy's expected payoff, compare e^* to any allocation $\bar{e} \in T_i^e$. Enemy's realized payoffs will be the same unless Blotto plays in T_i^b or T_{i+1}^b. If Blotto plays certain allocations in T_i^b (allocations in j in the Figure) Enemy will win on Battlefield 1 with e^* when he would have lost with $\bar{e}$. The trade-off is that if Blotto plays certain allocations in T_{i+1}^b, (allocations in m in the picture) Enemy will lose on Battlefield 2 with e^* when he would have won with $\bar{e}$. In order for Enemy to not have any payoff improving deviations the added probability of losing Battlefield 2 must be at least as great as the added probability of winning

[15] We are not discussing any of the larger triangles in the graph which contain smaller shapes (e.g. the axes and the resource constraints form triangles, but these are not what we are interested in). The triangles we are discussing are empty in Figure 5c

[16] Whether the bounds of the various T_i's are open or closed is left for Section 5.

Figure 6: Region 3 Mass Restrictions



(a) (b)

on Battlefield 1. In terms of Figure 6a this requires that $\mu_B(m) \geq \mu_B(j)$. This type of condition must be satisfied for any deviating full expenditure e^*. Figure 6b graphically demonstrates the similar restrictions on Enemy's randomization.

We now have a graphical method for depicting the complete set of equilibrium strategies for Blotto and Enemy. For each player it is the set of strategies where they allocate within each of their T_i's with probability $\frac{1}{n}$, and randomize within the T_i's in such a way that their opponent has no full expenditure, expected payoff improving deviations. We provide a graphical algorithm for constructing the T_i's and for visualizing the deviation preventing condition. The complete set of equilibria is the set of pairs of Blotto and Enemy equilibrium strategies and is given formally in Section 5.

4. Applications

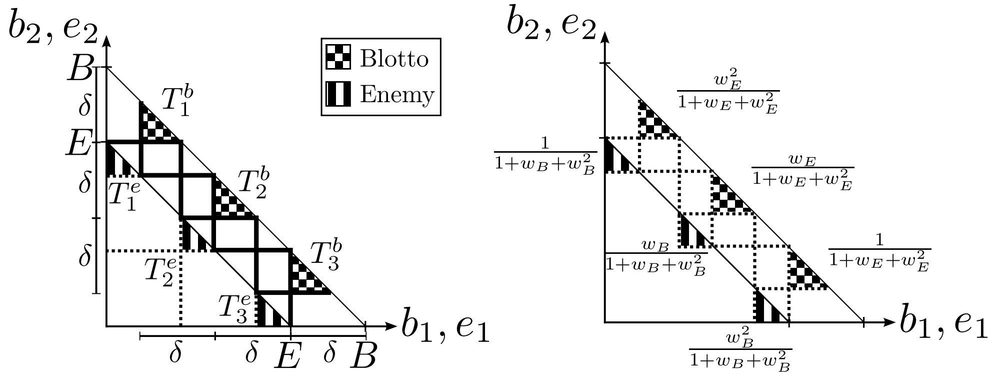
In this section we demonstrate the ability of our algorithm to aid further Blotto research. Specifically, we use the algorithm to find the complete set of equilibria for two previously unsolved generalizations of Gross and Wagner's (1950) original game. In this section we again just discuss intuition and leave the formal characterization for Section 5.

4.1. Unique Battlefield Values

Now we relax our earlier assumption that both players care about both battlefields equally. The game remains the same except for the following modification. If Blotto or Enemy wins Battlefield i they receive a payoff there of a_i^b or a_i^e, respectively. Both players will face the same constraints as before, but their objective functions are now different. Blotto's objective is now:

$$\max_{b_1, b_2} \left(\sum_{i=1}^{2} \left(Prob(b_i \geq e_i | \mu_E) \cdot a_i^b \right) \right)$$

Figure 7: Weights



(a) (b)

while now Enemy's is:

$$\max_{e_1,e_2}\left(\sum_{i=1}^{2}(Prob(e_i > b_i|\mu_B)\cdot a_i^e)\right).$$

To simplify the analysis, normalize the weight each player places on Battlefield 1 to 1 ($a_1^b = a_1^e = 1$). Then, let $w_B > 0$ and $w_E > 0$ be the relative weight Blotto and Enemy, respectively, place on Battlefield 2 ($w_B \equiv \frac{a_2^b}{a_1^b}$ and $w_E \equiv \frac{a_2^e}{a_1^e}$). To our knowledge, no equilibrium to these Blotto games have been found before.[17] However, we can characterize a large set of equilibria with a small extension of our graphical method.[18]

While we explain how to find our set of equilibrium strategies in a generic region, Figure 7 shows how our process applies to Region three. We follow the process in Section 3.3 to find the T_i's over which players may randomize (Figure 7a). Previously, player's would play in each T_i with probability $\frac{1}{n}$. Now that the players place different weights on the two battlefields, they need to play in each T_i with a different probability in order to make their opponent indifferent between his own T_i's.

Suppose Enemy is examining the expected payoff of playing in some T_i^e compared to playing in T_{i+1}^e (e.g. T_1^e v.s. T_2^e in Figure 7a). The realized payoff of playing in either will be the same unless Blotto plays in T_i^b or T_{i+1}^b. Moving from T_{i+1}^e to T_i^e allows Enemy to now win Battlefield 2 if Blotto plays in T_{i+1}^b but at the cost of now losing on Battlefield 1 when Blotto plays in T_i^b. Since Enemy values Battlefield 2 w_E times as much as Battlefield 1, his added

[17]Except in the constant sum case where $w_B = w_E$ in Gross and Wagner (1950).

[18]In fact, it seems as though the set we characterize is indeed complete, but we leave the proof of this fact for future work.

chance of losing Battlefield 1 needs to be w_E times his added chance of winning Battlefield 2. This implies that $\mu_B(T_i^b) = w_E \cdot \mu_B(T_{i+1}^b)$. As this must hold for all i:

$$\mu_B(T_i^b) = \frac{w_E^{n-i}}{\sum_{j=0}^{n-1} w_E^j},$$

as shown in Figure 7b.

We can also perform a similar analysis to find the the probability Enemy will play in each T_i^e. Suppose now that Blotto is examining the expected payoff of playing in some T_{i+1}^b compared to playing in T_i^b (e.g. T_2^b v.s. T_1^b in Figure 8a). The realized payoff of playing in either will be the same unless Enemy plays in T_i^e or T_{i+1}^e. Moving from T_i^b to T_{i+1}^b allows Blotto to now win Battlefield 1 if Enemy plays in T_{i+1}^e but at the cost of now losing on Battlefield 2 when Enemy plays in T_i^e. Since Blotto values Battlefield 2 w_B times as much as Battlefield 1, his added chance of winning Battlefield 1 needs to be w_B times his added chance of losing Battlefield 2. This implies that $\mu_E(T_{i+1}^e) = w_B \cdot \mu_E(T_i^e)$. As this must hold for all i:

$$\mu_E(T_i^e) = \frac{w_B^{i-1}}{\sum_{j=0}^{n-1} w_B^j},$$

as shown in Figure 7b.

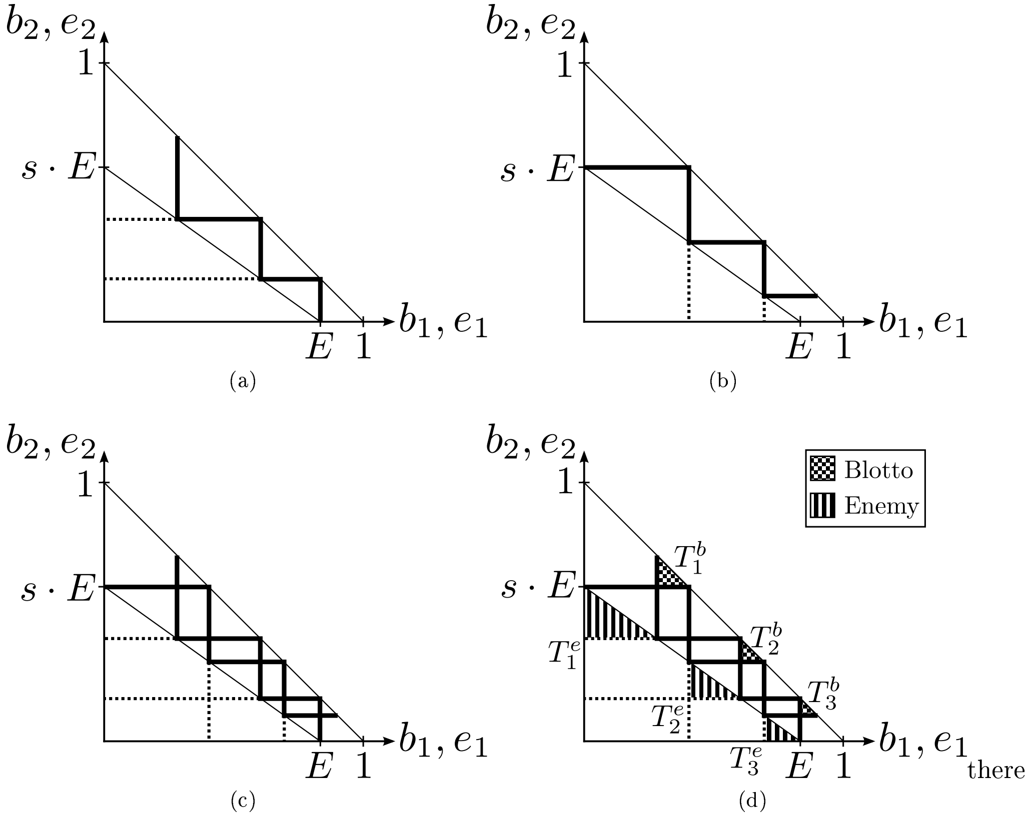
We will still need to place additional restrictions on how player's can randomize within their T_i's. However, we can show the restrictions in the same manner as before in Figure 6. The only difference is that now we need to account for the different weights. This is easily reconciled. In terms of Figure 6a the restriction now becomes $\mu_B(m) \geq w_E \cdot \mu_B(j)$ and similarly for μ_E. Our set of equilibrium in these non-constant sum Blotto games is going to be the set of pairs of Blotto and Enemy strategies that satisfy the conditions we've just described.

4.2. Force Effectiveness

We may wish to consider Blotto games where one player's forces have an advantage on one of the battlefields. For instance, suppose Blotto's forces are marines, while Enemy's are general infantry and attacking Battlefield 2 involves a water landing. We would expect that adding forces to Battlefield 2 will increase Blotto's strength there more than a similar increase by Enemy. As far as we know, there are no prior solutions to this generalization of the Blotto game, but our graphical method allows us to address it quite easily.

In order to deal with this case we need redefine a few parts of the game. The winner of a battlefield is no longer contingent only upon who has the *largest* force, but on who has the *strongest* force. So, we recast the problem as one of choosing a strength of force on each battlefield. First, consider Blotto and Enemy both committing all their forces to Battlefield 1. Without loss of generality, assume this means Blotto has a stronger force on Battlefield 1. Normalize

Figure 8: Force Effectiveness

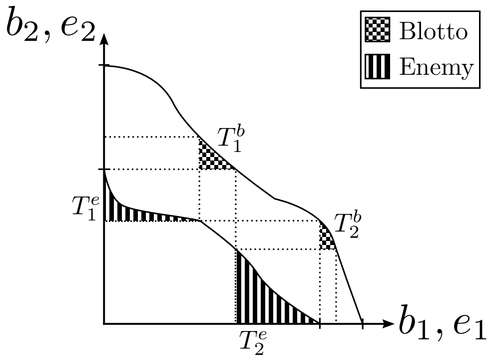


the strength of Blotto's force there to 1, and that of Enemy to E. We continue assuming that $E < 1$.

Since a player's forces may now be better at attacking either battlefield, each player will face their own trade-off when moving forces from one battlefield to the other. For now consider cases where each player can trade strength between the two battlefields linearly. So we normalize the rate at which Blotto can trade strength on Battlefield 1 for strength on Battlefield 2 to 1 and let Enemy trade off strength on Battlefield 1 for strength on Battlefield 2 at rate s. So, if Blotto and Enemy sent all their forces to Battlefield 2, their strengths there would be 1 and $s \cdot E$, respectively. These trade-offs are reflected by the constraints in Figure 8.

Our previous graphical algorithm described in Section 3.3 works here without modification. We only need to use the new budget constraints. Figure 8 demonstrates how our method works here in the same way as Figure 5 did for the case where forces were equally effective on all battlefields. If we wanted allow for this modification and general battlefield weights, the analysis from

Figure 9: Generic Budget Constraints



Section 4.1 would carry over directly.

Note that if Blotto's forces were relatively ineffective when attacking Battlefield 2 the resource constraints could cross and our method would no longer work. However, in this case there is a very simple Nash equilibrium. Both players can "hunker down" and send their full force to the battlefield where they are the stronger and guarantee victory there.

In fact, the assumption of linear budget constraints isn't necessary. Our algorithm appears to be able to deal with any strictly decreasing budget constraints (so long as Enemy's budget constraint is completely within Blotto's and Enemy's budget constraint intersects both axes). Figure 9 shows how our graphical algorithm could be applied in these cases. In Section 5 we will formally describe our set of equilibrium in this general case.

5. Again, With More Rigor

Here we generalize and formalize the above. Specifically, we consider the following very generic two battlefield Blotto game: The two players simultaneously set $\{b_i\}_{i=1}^2$ and $\{e_i\}_{i=1}^2$, denoting Blotto and Enemy's respective strength of force on Battlefield i. If Blotto has a (weakly) stronger force on Battlefield i he wins that battle and receives a payoff on that front of $a_i^b > 0$. Otherwise, Enemy wins that battle and receives $a_i^e > 0$. The losing player receives a payoff of 0 from that front. Players attempt to maximize the expected sum of their payoffs across the fronts. We denote Blotto's or Enemy's strategy as μ_B or μ_E, respectively. Blotto's allocation is constrained by $b_1 \geq 0$, $b_2 \geq 0$ and a generic resource constraint, $b_2 \leq f(b_1)$. So, his optimization problem is:

$$\max_{b_1,b_2} \left(\sum_{i=1}^{2} \left(Prob(b_i \geq e_i | \mu_E) \cdot a_i^b \right) \right) \quad s.t. \tag{1}$$

$$b_1 \geq 0,\ b_2 \geq 0\ b_2 \leq f(b_1) \tag{2}$$

Similarly, Enemy is constrained by $e_1 \geq 0$, $e_2 \geq 0$, and $e_2 \leq g(e_1)$ and his optimization problem is:

$$\max_{e_1,e_2} \left(\sum_{i=1}^{2} (Prob(e_i > b_i|\mu_B) \cdot a_i^e) \right) \quad s.t. \tag{3}$$

$$e_1 \geq 0,\ e_2 \geq 0,\ e_2 \leq g(e_1). \tag{4}$$

To make the problem easier we make the following assumptions. Without loss of generality normalize $a_1^b = a_1^e = 1$. Now we only care about relative weight each player places on Battlefield 2. So, let $w_b = \frac{a_2^b}{a_1^b}$ and $w_e = \frac{a_2^e}{a_1^e}$. Also assume that

$$\exists\ B_1, B_2 > 0\ s.t.\ f(B_1) = 0,\ f(0) = B_2. \tag{5}$$

$$\exists\ E_1, E_2 > 0\ s.t.\ g(E_1) = 0,\ g(0) = E_2. \tag{6}$$

In other words, there exist x and y intercepts, E_1 and E_2 (B_1 and B_2), of Enemy's (Blotto's) resource constraint.[19] Furthermore, assume that both f and g are continuous, strictly decreasing functions. Finally, we need to make an assumption so that Blotto is always the advantaged player. We do this by assuming that there is some minimum amount by which Blotto's budget constraint is always higher than Enemy's[20] or

$$\exists \epsilon > 0\ s.t.\ \forall x_1 \in [0, E_1]\ f(x_1) \geq g(x_1) + \epsilon \tag{7}$$

Also, we will use the following to composite functions:

$$h(x) \equiv g^{-1}(f(x)). \tag{8}$$

$$p(x) \equiv g(f^{-1}(x)) \tag{9}$$

Consider a generic two battlefield Blotto game in Region n.[21] Specifically,[22]

$$E_2 \in (h^{n-2}(f(E_1)), h^{n-1}(f(E_1))]. \tag{10}$$

Also define the following sets of allocations:[23]

$$\forall i = 1, ..., n \quad T_i^b \equiv \{(b_1, b_2) :\ (b_1 \geq h^{n-i}(E_1),\ b_2 \geq p^{i-1}(E_2),\ b_2 \leq f(b_1)\} \tag{11}$$

[19] We could relax this assumption somewhat for Blotto. However, it seems realistic and makes the proof cleaner.

[20] This rules out the possibility that the budget constraints cross, but often such games have simple equilibria. For instance, if the budget constraints cross exactly once (or any odd number of times), each player can guarantee victory on exactly one battlefield. The simple Nash is for each player to "hunker-down" and always send an unbeatable force to the front on which they can guarantee victory.

[21] Any generic Blotto game must be in some such region. Specifically since $f(x) \geq g(x) + \epsilon$, our graphical algorithm will find $n \leq \left\lceil \frac{E_2}{\epsilon} \right\rceil$.

[22] The conditions for Region 1, the trivial region, are slightly different. There, $E_2 \leq f(E_1)$

[23] We follow the standard convention where $f^0(x) = x$.

$$\forall i=2,3,...,n-1 \quad T_i^e \equiv \{(e_1,e_2):\ (e_1 > f^{-1}\left(p^{i-2}\left(E_2\right)\right),\ e_2 > f\left(h^{n-i-1}\left(E_1\right)\right),\ e_2 \le g(e_1))\} \tag{12}$$

$$T_1^e \equiv \{(e_1,e_2):\ (e_1 \ge 0,\ e_2 > f\left(h^{n-2}\left(E_1\right)\right),\ e_2 \le g(e_1))\} \tag{13}$$

$$T_n^e \equiv \{(e_1,e_2):\ (e_1 > f^{-1}\left(p^{n-2}\left(E_2\right)\right),\ e_2 \ge 0,\ e_2 \le g\left(e_1\right))\} \tag{14}$$

The above sets correspond to the triangles we found in the prior sections which bounded the players' equilibrium allocations. Since we are allowing nonlinear budget constraints these sets may no longer correspond to triangles, but we use the notation T as is corresponds directly to the simple case. We also need to define some sets that so that we can restrict how players can randomize over these "triangles" (as we did in the prior sections). $\forall i < 1,2,...,n-1, \forall x \in \mathbb{R}$:

$$j_b^{x,i} \equiv \{(b_1,b_2):\ ((b_1,b_2) \in T_i^b,\ b_1 < x)\} \tag{15}$$

$$k_b^{x,i} \equiv \{(b_1,b_2):\ ((b_1,b_2) \in T_{i+1},\ b_2 \ge g\left(x\right))\} \tag{16}$$

$$j_e^{x,i} \equiv \{(e_1,e_2):\ ((e_1,e_2) \in T_{i+1}^e,\ e_1 \le x)\} \tag{17}$$

$$k_e^{x,i} \equiv \{(e_1,e_2):\ ((e_1,e_2) \in T_i^e,\ e_2 > f\left(x\right))\} \tag{18}$$

If Enemy were to play $(x, g(x))$, $j_b^{x,i}$ represents the portion of T_i^b where Blotto would lose on Battlefield 1. While $k_b^{x,i}$ represents the portion of T_{i+1} where Blotto would win on Battlefield 2. Conversely, if Blotto were to play $(x, f\left(x\right))$, $j_e^{x,i}$ represents the portion of T_{i+1}^e where Blotto would win on Battlefield 1. While $k_e^{x,i}$ represents the portion of T_i^e where Blotto would lose on Battlefield 2.

Now, define Ω^B as the set of probability measures, μ_B, which satisfy the following two properties:

Property 1b) $\forall i = 1,...n$

$$\mu_B(T_i^b) = \frac{w_E^{n-i}}{\sum\limits_{j=0}^{n-1} w_E^j}$$

Property 2b) $\forall i < 1,2,...,n-1, \quad \forall x \in [h^{n-i}(E_1), f^{-1}\left(p^{i-2}\left(E_2\right)\right)]$:

$$\mu_B(j_b^{x,i}) - \mu_B(k_b^{x,i}) \cdot w_E \le 0$$

Now, define Ω^E as the set of probability measures, μ_E, with the following two properties:

Property 1e) $\forall i = 1,...,n$

$$\mu_E(T_i^e) = \frac{w_B^{i-1}}{\sum\limits_{j=0}^{n-1} w_B^j}$$

Property 2e) $\forall i = 1,2,...,n-1\ \forall x \in (f^{-1}\left(p^{i-1}\left(E_2\right)\right), h^{n-i-1}(E_1))$

$$\mu_E(j_e^{x,i}) - \mu_E(k_e^{x,i}) \cdot w_B \le 0$$

These definitions lead to two Theorems which we prove in Appendix Appendix A:

Theorem 1. *Any pair of strategies* $\{\mu_B, \mu_E\}$ *such that* $\mu_B \in \Omega^B$ *and* $\mu_E \in \Omega^E$ *form a Nash equilibrium to the two battlefield Colonel Blotto game.*

In other words if you pair a Blotto strategy from Ω^B, with an Enemy strategy from Ω^E, you will have formed a Nash equilibrium.

Theorem 2. *When* $w_B = w_E$ *the complete set of Nash Equilibrium to any two battlefield Colonel Blotto game is the set of pairs* (μ_B, μ_E) *such that* $\mu_B \in \Omega^B$ *and* $\mu_E \in \Omega^E$.

While we wait to prove Theorems 1 and 2 until Appendix Appendix A, the underlying conditions on the Ω's correspond quite directly to the conditions we found graphically in Sections 3 and 4. Property 1b(e) simply requires that Blotto (Enemy) randomizes over the areas we found in Figures 5d, 8, and 9 and that he plays within each area with the probabilities found in Section 4.1. Property 2b(e) simply requires that Blotto (Enemy) distributes his mass over the areas from Property 1b(e) in such a way that he does not provide his opponent with any profitable deviations. This condition reflects the requirements we found graphically in Figures 4a and 6a (4b and 6b).

6. Extensions

Experimentally, Chowdhury and Sheremeta (2009) are able to confirm most of Roberson's (2006) theoretical predictions. Prior to our work, the literature only knew of a finite number of equilibria to any two battlefield Blotto game. This would have made testing two battlefield Blotto predictions (experimentally or empirically) problematic. The set of Nash equilibrium strategies we find is infinite. Even if one used data that came from Nash equilibrium two battlefield Blotto play, there would have been no reason to assume that the data came from one of the previously known equilibria.

For a larger number of battlefields ($m \geq 3$) our analysis would seem to apply most directly to the region where Roberson (2006) only provided example equilibria. We refer to this region, where $B \in [(m-1)E, mE]$, as "very asymmetric." Specifically, consider the case where $B \in [(m-1/2)E, mE)$. In relation to our above analysis, this would be analogous to Region 2 where $E > \delta \geq \frac{1}{2}E$; Blotto can guarantee himself victory on $(m-1)$ battlefields, and give Enemy only one shot at attacking the less defended battlefield. Blotto can randomly choose one battlefield and send $\frac{1}{2}E$ units there, and send E units to the rest. The best Enemy can then do is randomly choose a front and attack it with full force.[24] We have yet to characterize the complete set of equilibrium in this case, but it

[24]If the fronts have different values to different players, as in Section 4.1, players could weight their randomizations to make each other indifferent. As far as we know this would be a new equilibrium to a previously unsolved Blotto game.

seems as though it will satisfy properties similar to 1b(e) and 2b(e). Analogous to property 1e) Enemy will randomize which battlefield he attacks heavily, and analogous to property 2e) will be a set of conditions that ensure Blotto does not wish to deviate.

7. Conclusion

We provide a graphical algorithm for constructing the set of all Nash equilibrium strategies to any canonical two battlefield Colonel Blotto game. Furthermore, we provide a formal definition of these sets and prove their completeness. Our algorithm takes a very technical game and makes it far more accessible. We are able to start with the simple logic of the trivial case and show how that logic extends to much more complicated versions of the canonical game. In fact, our algorithm demonstrates further utility; it applies easily to two previously unsolved generalizations of the game.

Our work could prove useful for further empirical or theoretical work. While the set of equilibria we find is infinite, our characterization provides some easily testable criteria: In equilibrium each player should only play allocations from one of n distinct areas. We define these areas and provide the probability with which they should play in each. We have already shown how to our graphical algorithm can advance theoretical work on Blotto games. We used it to easily characterize a set of equilibria to two previously unsolved generalizations of the canonical Blotto game. It also seems that work on "very asymmetric" Blotto games with three or more battlefields may be able to use a higher dimensional version of our algorithm.

Characterizing the complete set of equilibrium strategies to the canonical Blotto game exposes new equilibrium and the full set of conditions constraining them. Interestingly, there exist equilibria where neither player fully expends their resources, even though there is no payoff to unallocated resources (perhaps this suggests possible equilibrium refinement techniques or potential insight into the shadow value of the players' resources). Additionally, extending the logic of the equilibrium construction algorithm begins to yield insight into more complicated variants of the game, which may be more representative of real military, political, or academic environments.

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Appendix A. Proofs of Theorems 1 and 2

First we prove that all pairs of strategies from Ω^B and Ω^E constitute a Nash Equilibrium. We then show that in the constant sum case ($w_B = w_E$) no other strategies are a part of any Nash Equilibrium. For the sake of readability start with several definitions. First, define two projection operators. $\Psi_1(S)$ is the set of all scalars that are the first dimension of some two dimensional point in the set of two dimensional points, S.

$$\Psi_1(S) \equiv \{x_1 : \ (\exists x_2 \in \mathbb{R} \ s.t. \ (x_1, x_2) \in S)\}$$

$\Psi_2(S)$ is defined similarly:

$$\Psi_2(S) \equiv \{x_2 : \ (\exists x_1 \in \mathbb{R} \ s.t. \ (x_1, x_2) \in S)\}$$

[25]We also define the set of all points in some T_i^b (T_i^e):

$$T^b \equiv T_1^b \cup T_2^b \cup ... \cup T_n^b$$

$$T^e \equiv T_1^e \cup T_2^e \cup ... \cup T_n^e$$

Finally, we define the sets of all feasible Blotto and Enemy allocations:

$$F^b \equiv \{(b_1, b_2) : \ b_1, b_2 \geq 0 \ and \ b_2 \leq f_b(b_1)\}$$

$$F^e \equiv \{(e_1, e_2) : \ e_1, e_2 \geq 0 \ and \ e_2 \leq f_e(e_1)\}.$$

The following Lemma gives us the intervals of battlefield allocations within the various T_i^x's ($\forall x = b, e$).

Lemma 3. $\forall i = 1, 2, ..., n$:

$$\Psi_1(T_i^e) = \left(f^{-1}\left(p^{i-2}(E_2)\right), h^{n-i}(E_1)\right) \tag{A.1}$$

[26]

$$\Psi_1(T_i^b) = \left[h^{n-i}(E_1), f^{-1}\left(p^{i-1}(E_2)\right)\right] \tag{A.2}$$

$$\Psi_2(T_i^e) = \left(f\left(h^{n-i-1}(E_1)\right), p^{i-1}(E_2)\right) \tag{A.3}$$

[27]

$$\Psi_2(T_i^b) = \left[p^{i-1}(E_2), f\left(h^{n-i}(E_1)\right)\right] \tag{A.4}$$

[25]For instance, if $S = \{(1,3), (2,5)\}$, then $\Psi_1(S) = \{1,2\}$ and $\Psi_2(S) = \{3,5\}$.

[26]For $i = 1$, the lower bound becomes $f^{-1}\left(p^{-1}(E_2)\right) = f^{-1}\left(f\left(g^{-1}(E_2)\right)\right) = 0$ and is actually a closed boundary. For $i = n$, the upper boundary becomes $h^0(E_1) = E_1$ and is actually a closed boundary.

[27]For $i = 1$, the upper bound becomes $p^0(E_2) = E_2$ and is actually a closed boundary. For $i = n$, the lower boundary becomes $f\left(h^{-1}(E_1)\right) = f\left(f^{-1}(g(E_1))\right) = 0$ and is actually a closed boundary.

We will simultaneously prove Lemma 3 with our proof of the next lemma. Informally and abusing notation, we can rewrite the interval $\left[0, f^{-1}\left(p^{n-1}\left(E_2\right)\right)\right]$ as

$$[\Psi_1(T_1^e), \Psi_1(T_1^b), \Psi_1(T_2^e), \Psi_1(T_2^b), ..., \Psi_1(T_n^e), \Psi_1(T_n^b)]$$

or we could write the interval $\Psi_2(T^e) \cup \Psi_2(T^b) = \left[0, f\left(h^{n-1}\left(E_1\right)\right)\right]$ as

$$[\Psi_2(T_n^e), \Psi_2(T_n^b), \Psi_2(T_{n-1}^e), \Psi_2(T_{n-1}^b), ..., \Psi_2(T_1^e), \Psi_2(T_1^b)].$$

.

Lemma 4. *Formally,* $\Psi_1(T^e) \cup \Psi_1(T^b) = \left[0, f^{-1}\left(p^{n-1}\left(E_2\right)\right)\right]$ *and* $\Psi_2(T^e) \cup \Psi_2(T^b) = \left[0, f\left(h^{n-1}\left(E_1\right)\right)\right]$ *while* $\Psi_1(T^e) \cap \Psi_1(T^b) = \{E_1\}$ and $\Psi_2(T^e) \cap \Psi_2(T^b) = \{E_2\}$.

Proof. Refer back to equations 13-14. Consider the bounds for any $e_1 \in \Psi_1(T_i^e)$. Its open[28] infimum is $f^{-1}\left(p^{i-2}\left(E_2\right)\right)$. Changing the two other constraints on T_i^e to equalities and solving we find that for $e_1 \in \Psi_1(T_i^e)$ the open[29] supremum is $h^{n-i}(E_1)$, which is the closed infimum of $b_1 \in \Psi_1(T_i^b)$. Similar algebra for the other relevant bounds in Lemma 3 combined with simple induction confirms Lemmas 3 and 4. □

Remark 5. $\forall i \in \{1, ..., n\}$ any $e_1 \in \Psi_1(T_i^e)$ is strictly less than any $b_1 \in \Psi_1(T_i^b)$ which is strictly less than any $e_1 \in \Psi_1(T_{i+1}^e)$.[30] Also, $\forall i \in \{1, ..., n\}$ any $e_2 \in \Psi_2(T_{i-1}^e)$ is strictly greater than any $b_2 \in \Psi_2(T_i^b)$ which is strictly greater than any $e_2 \in \Psi_2(T_i^e)$.[31] Formally,

$$\forall i \in \{1, ..., n-1\} \quad (e_1^i \in \Psi_1(T_i^e),\ e_1^{i+1} \in \Psi_1(T_{i+1}^e),\ b_1^i \in \Psi_1(T_i^b)) \implies (e_1^i < b_1^i < e_1^{i+1}) \tag{A.5}$$

$$(e_1^n \in \Psi_1(T_n^e), b_1^n \in \Psi_1(T_n^b)) \implies e_1^n \leq b_1^n \tag{A.6}$$

$$\forall i \in \{2, 3, ..., n\} \quad (e_2^{i-1} \in \Psi_2(T_{i-1}^e),\ e_2^i \in \Psi_2(T_i^e),\ b_2^i \in \Psi_2(T_i^b) \implies (e_2^i < b_2^i < e_2^{i-1}) \tag{A.7}$$

$$(e_2^1 \in \Psi_2(T_1^e), b_2^1 \in \Psi_2(T_2^e)) \implies e_2^1 \leq b_2^1 \tag{A.8}$$

Remark 5 follows directly by examining the bounds in Lemma 3.

Appendix A.1. Proof of Theorem 1

In this section we will prove that satisfying our characterization is a sufficient condition for Nash equilibrium. In other words, any pair of strategies we've characterized in fact forms a Nash Equilibrium to this game. Before proceeding with the formal proof we provide the intuition. Properties 1b and 1e specify that in any equilibrium Blotto and Enemy each randomize over n distinct areas

[28] closed when $i = 1$

[29] closed when $i = n$

[30] The former inequality is weak when $i = n$. Obviously, we ignore the latter when $i = n$.

[31] The later inequality is weak when $i = 1$. Obviously, we ignore the former when $i = 1$.

($T_1^b, ..., T_n^b$ and $T_1^e, ..., T_n^e$). Blotto and Enemy's potential equilibrium allocations on either battlefield only overlap at one point in the following sense:

$$\Psi_1(T^b) \cap \Psi_1(T^e) = \{E_1\},$$
$$\Psi_2(T^b) \cap \Psi_2(T^e) = \{E_2\}.$$

Given that ties always go to Blotto, we can calculate players' expected payoffs .[32] When they both play strategies satisfying Properties 1b and 1e, Blotto achieves an expected payoff of $\frac{\sum_{j=0}^{n} w_B^j}{\sum_{j=0}^{n-1} w_B^j}$ while Enemy earns $\frac{\sum_{j=1}^{n-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j}$. Given these payoffs, Property 2b(e) ensures that Enemy (Blotto) has no full expenditure allocation outside the T_i^e's (T_i^b's) which provide a payoff strictly greater than $\frac{\sum_{j=1}^{n-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j}$ ($\frac{\sum_{j=0}^{n} w_B^j}{\sum_{j=0}^{n-1} w_B^j}$). Since all allocations in the players' supports provide the same payoff, and there exist no allocations providing higher payoffs, pairs of strategies from these distributions constitute a Nash equilibrium.

Proposition 6. *Any pair of strategies satisfying properties 1b, 1e, 2b, and 2e constitutes a Nash Equilibrium to the two battlefield Colonel Blotto Game in region n.*

Proof. This Proposition is simply an alternative way of stating Theorem 1. Given Remark 5 and Property 1b, we know that against any Blotto strategy from our definition, when Enemy plays in T_i^e his probability of winning on Battlefield 1 is:

$$\mu_B(T_1^b \cup ... \cup T_{i-1}^b) = \frac{\sum_{j=n-(i-1)}^{n-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j}$$

and his probability of winning Battlefield 2 is:

$$\mu_B(T_{i+1}^b \cup ... \cup T_n^b) = \frac{\sum_{j=0}^{n-i-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j}.$$

The total expected payoff is then:

$$1 \cdot \frac{\sum_{j=n-(i-1)}^{n-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j} + w_E \cdot \frac{\sum_{j=0}^{n-i-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j} = \frac{\sum_{j=1}^{n-1} w_E^j}{\sum_{j=0}^{n-1} w_E^j} \tag{A.9}$$

for any allocation in any T_i^e. Similarly, against any Enemy strategy from above, when Blotto plays in T_i^b his probability of winning Battlefield 1 is:

$$\mu_E\left(T_1^e \cup ... \cup T_i^e\right) = \frac{\sum_{j=0}^{i-1} w_B^j}{\sum_{j=0}^{n-1} w_B^j}$$

[32] If we instead assumed that ties were decided by a coin flip, we would come up with nearly the same equilibria. However, ties would no longer be possible, Blotto's T_i^b's would have open boundaries, Enemy's T_i^e's would have only closed boundaries, and the open and closed boundaries of the regions would be swapped.

and his probability of winning Battlefield 2 is:

$$\mu_E\left(T_i^e \cup ... \cup T_n^e\right) = \frac{\sum_{j=i-1}^{n-i} w_B^j}{\sum_{j=0}^{n-1} w_B^j}$$

The total expected payoff is then

$$1 \cdot \frac{\sum_{j=0}^{i-1} w_B^j}{\sum_{j=0}^{n-1} w_B^j} + w_B \cdot \frac{\sum_{j=i-1}^{n-i} w_B^j}{\sum_{j=0}^{n-1} w_B^j} = \frac{\sum_{j=0}^{n} w_B^j}{\sum_{j=0}^{n-1} w_B^j} \tag{A.10}$$

for any allocation in any T_i^b.

We now show that there are no allocations for Enemy or Blotto that provide a strictly higher expected payoff than we found in the previous paragraph. Note that if either player were to have an expected payoff improving deviation from the strategies we defined, they must have a full expenditure payoff improving deviation.[33] Therefore, we only need to show that there are no payoff improving full expenditure deviations. So, we check full expenditure deviations outside of any T_i^e or T_i^b.

Consider a generic full expenditure Enemy deviation (e_1^*, e_2^*).[34] Given that $(0, E_2)$ is in T_1^e and $(E_1, 0)$ is in T_n^e, (e_1^*, e_2^*) must lie "between" some T_i^e and T_{i+1}^e.[35] Let (e_1, e_2) be a non-deviating allocation in T_i^e. Examine Property 2b with $x = e_1^*$. The realized payoff to Enemy of playing (e_1^*, e_2^*) against any of our Blotto strategies will be the same as if he had played (e_1, e_2) unless Blotto plays in T_i^b or T_{i+1}^b. If Blotto plays in T_i^b the deviant strategy *may* do better[36] on Battlefield 1 (without changing the outcome on Battlefield 2). The cost is that if Blotto plays in T_{i+1}^b the deviant strategy may do worse on Battlefield 2 (without changing the outcome on Battlefield 1). Using the notation of Property 2b, any b_1 in $j_b^{e_1^*,i}$ will lose to e_1^* (while it would have beat e_1) and any b_2 in $k_b^{e_1^*,i}$ will beat e_2^* (while it would have lost to e_2). Property 2b then says that by moving from any (e_1, e_2) in T_i^e to (e_1^*, e_2^*) the additional probability of winning on Battlefield 1 is weakly less than the additional probability of losing on Battlefield 2 times the weight placed on that battlefield. Therefore, no full expenditure deviation (e_1^*, e_2^*) is payoff improving, and therefore no deviation is payoff improving.

The same line of reasoning applies directly to Property 2e and full expenditure deviations which lie "between" some T_i^b and T_{i+1}^b.[37] Additionally, there are full expenditure deviations which do not lie "between" some T_i^b and some T_{i+1}^b.[38]

---

[33] As the expected payoff must be weakly increasing in expenditure on either battlefield.

[34] Clearly $e_2^* = g(e_1^*)$.

[35] By "between" we mean: $\forall e_1^i \in \Psi_1(T_i^e), e_1^{i+1} \in \Psi_1(T_{i+1}^e) \quad e_1^i < e_1^* < e_1^{i+1}$, and similarly for e_2^*.

[36] By "do better" we mean e_1^* would be larger than Blotto's Battlefield 1 allocation, whereas e_1 would be weakly less.

[37] Specifically, Property 2e ensures that a full expenditure deviating allocation by Blotto, (b_1^*, b_2^*), cannot be payoff improving. Simply set $b_1^* = x$ in the property and the same line of reasoning follows.

[38] For instance, $(B, 0)$ and $(0, B)$.

Specifically, there are two more deviating types of full expenditure allocations: a (b_1^*, b_2^*) where $\forall(b_1, b_2) \in T_1^b \quad b_1^* < b_1 \quad and \quad b_2^* > b_2$ or a $(b_1^\#, b_2^\#)$ where $\forall(b_1, b_2) \in T_n^b \quad b_1^\# > b_1 \quad and \quad b_2^\# < b_2$.[39] In the former, Blotto increases allocations to Battlefield 2 at the expense of Battlefield 1, relative to T_1^b. However, in T_1^b, Blotto is guaranteeing victory on Battlefield 2, so this can not be payoff improving. Similar logic applies to the later type of allocations.

Thus, if Blotto plays $\mu_B \in \Omega^B$ and Enemy plays $\mu_E \in \Omega^E$, they would both be playing best responses to the other's strategy. Therefore any such pair $\{\mu_B, \mu_E\}$ constitutes a Nash equilibrium. □

Appendix A.2. Proof of Theorem 2

We now prove that in the constant sum case there are no other strategies which could be part of a Nash equilibrium. Specifically, we consider the version of the Blotto game from Section 5 while assuming $w = w_B = w_E$. Before proceeding we need the following definition and lemma:[40]

Definition 7. A game is said to feature *Constant Payoffs* if, for each player, the expected payoff is the same in all equilibria.

Our constant-sum Blotto games features Constant Payoffs as a direct application of Sion's (1958) general minimax theorem.[41]

Lemma 8. *(Equilibrium Interchangeability)*
In any two-player, constant-sum game which features Constant Payoffs, every strategy from any equilibrium is a best response to any opponent strategy from any (other) equilibrium.

Lemma 8 follows quite directly from Constant Payoffs and is proven in Appendix Appendix C.[42] Equilibrium Interchangeability allows us to consider equilibrium strategies for Blotto and Enemy separately. Unlike with most multiple equilibria games, there is no need to worry about pairing with a particular opponent equilibrium strategy. This turns out to be quite a powerful tool. If we discover just one Nash Equilibrium (pair of strategies) all the remaining equilibrium are simply the cross of all the Blotto strategies that form an equilibrium with the one known Enemy strategy, and all the Enemy strategies that form an equilibrium with the one known Blotto strategy.

Our proof will be organized as follows. First of all, any strategy satisfying Property 1b(e) while violating property 2b(e) cannot be a part of any Nash

[39] Given Lemma 3, Remark 5 and Property 1b, there are no other full expenditure deviations.

[40] Vega-Redondo (2003, pp. 47-50) provides an excellent discussion of Constant Payoffs and Equilibrium Interchangeability in finite games.

[41] The relevant theorem in Sion (1958) is 3.4. Thanks to Brian Roberson for providing this reference.

[42] We are not the first to use these two properties when analyzing Blotto Games (e.g. see Roberson (2006)). However, we generally see weaker statements of Lemma 8. Therefore, we provide a proof for the sake of completeness, even though similar proofs already exist (e.g. see Vega-Redondo (2003, pp. 50)).

Equilibrium as it provides Enemy (Blotto) with an allocation whose payoff is strictly greater than his constant equilibrium payoff. Then we show that any strategy violating property 1b(e) does not form an equilibrium with specific Enemy (Blotto) equilibrium strategies. Therefore such strategies are not a part of any equilibrium by Equilibrium Interchangeability.

Proposition 9. *In the constant sum game* ($w = w_B = w_E$) *any Enemy strategy which is a part of some Nash Equilibrium is in* Ω^E.

Proof. Because of Equilibrium Interchangeability, all we need to show in order to prove that a strategy is not a part of any Nash Equilibrium is that the strategy does not form a Nash Equilibrium when paired with a strategy that we've already shown was a part of some Nash Equilibrium. We make use of the Blotto strategy μ_B^*, where in each T_i^b Blotto plays the allocations $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$ and $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$ with probability $\frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j}$ each .[43] Clearly, this means he plays all other allocations with probability zero.[44]

We prove Proposition 9 by contradiction. Suppose there exists a Nash Equilibrium Enemy strategy that is not in Ω^E. Such a strategy must then either violate Property 1e or satisfy Property 1e and violate Property 2e. In proving that all our strategies were indeed part of a Nash Equilibrium, we've already shown how a violation of Property 2e alone would provide Blotto with a payoff improving deviation, so we rule out that possibility. The only other way Proposition 9 could be false is if there were a Nash equilibrium Enemy strategy which violated property 1e. We divide deviations from property 1e into three possible cases. Figure A.10 provides a graphical reference (in Region 3) to aid the reader.

Deviation 1) Enemy could play over an area that sends less to both fronts than some (e_1, e_2) in some T_i^e. Formally, this would have Enemy play a $\mu_E^{d_1}(Y)$ such that the following three statements hold for some set of points S:

$$S \cap T^e = \emptyset$$

$$\mu_E^{d_1}(S) > 0$$

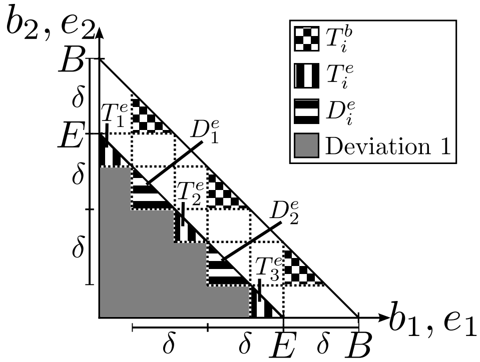
$$\forall (e_1^*, e_2^*) \in S, \quad \exists (e_1, e_2) \in T^e \quad s.t. \quad e_1^* \le e_1 \quad and \quad e_2^* \le e_2$$

The first condition implies that at least one of these inequalities in the third holds strictly. We already have a contradiction as this could not be a best response to μ_B^* which has Blotto playing the lower bounds of $e_1$and e_2 in T_i^e

[43] A simple algebraic or graphical analysis shows that these strategies are in Ω^B. These are the intersections of Blotto's resource constraint with the two other bounds on T_i^b.

[44] As $\sum_{i=1}^{n} 2 \cdot \frac{w^{n-i}}{2\cdot\sum_{j=0}^{n-1} w^j} = 1$.

Figure A.10: Enemy Deviations in Region 3



with positive probability. All $(e_1^*, e_2^*) \in S$ must then provide a strictly lower expected payoff than playing in some T_i^e.

This only leaves two possible types of deviations by Enemy: He could play with mass other than $\frac{w^{i-1}}{\sum_{j=0}^{n-1} w^j}$ over some T_i^e **(Deviation 3)** and/or he could play with mass over a region S[45] where $\forall (e_1^*, e_2^*) \in S, \quad \forall (e_1, e_2) \in T^e$ either:

$$e_1^* > e_1 \text{ } or \text{ } e_2^* > e_2$$

(Deviation 2). Given the bounds of the T_i^e's any such region S must be a within the set of points D_i^e, indexed by $i = 1, 2, \dots n-1$, where

$$D_i^e \equiv \left\{ (e_1, e_2) : \ (e_1 \geq h^{n-i}(E_1), \ e_2 \geq p^i(E_2), \ and \ e_2 \leq g\,(e_1)) \right\}.$$

Intuitively **Deviation 1** represents Enemy placing some mass on allocations which, relative to some T_i^e, always send less to one battlefield, without increasing the allocation to the other. **Deviation 3** has him playing an "incorrect" mass on some T_i^e. **Deviation 2** represents Enemy placing mass on allocations which, relative to any T_i^e always send less to one battlefield, but increase the allocation to the other.

We simultaneously, inductively prove that neither of the latter two deviations is possible. Consider a generic T_i^e and D_i^e and a deviating Enemy strategy μ_E^d that forms an Nash equilibrium with any $\mu_B \in \Omega^B$. Consider some $i \in \{1, 2, 3, ..., n-1\}$. Assume that

$$\forall j = 1, 2, \dots, i-1, \quad \mu_E^d(D_j^e) = 0 \ and \ \mu_E^d(T_j^e) = \frac{w_B^{i-1}}{\sum_{j=0}^{n-1} w_B^j} \tag{A.11}$$

In other words, there has not "yet" been a **Deviation 2** or **Deviation 3**.

[45] Again assume $S \cap T^e = \emptyset$

Note that $\forall e_2^* \in \left[0, g\left(h^{n-i}\left(E_1\right)\right)\right]$ it cannot be the case that $\left(\left(h^{n-i}\left(E_1\right)\right), e_2^*\right)$ is a best response to μ_B^*. Relative to any $(e_1, e_2) \in T_i^e$, $\left(\left(h^{n-i}\left(E_1\right)\right), e_2^*\right)$ has the same probability of winning on Battlefield 1,[46] but has a lower probability of winning on Battlefield 2.[47] Now, suppose the mass over $\mu_E^d(T_i^e) < \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$. Given equation A.11 and the fact that we've ruled out **Deviation 1**, when Blotto plays $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$,[48] he wins Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e\right) < \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ but still wins Battlefield 2 with probability $1 - \mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$ for a total expected payoff strictly less than $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$, which is Blotto's constant expected payoff in all equilibrium.

Similarly, if $\mu_E^d\left(T_i^e\right) > \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$ then when Blotto plays $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$, he wins Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e\right) > \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ but still wins Battlefield 2 with probability $1 - \mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$ for a total expected payoff strictly greater than $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$ again a contradiction. Therefore, $\mu_E^d(T_i^e) = \frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$.

Now suppose $\mu_E^d(D_i^e) > 0$. Now, when Blotto plays $\left(f^{-1}\left(p^{i-1}\left(E_2\right)\right), p^{i-1}\left(E_2\right)\right)$[49] he expects to win on Battlefield 1 with probability $\mu_E^d\left(T_1^e \cup ... \cup T_i^e \cup D_i^e\right) > \frac{\sum_{j=0}^{i-1} w^j}{\sum_{j=0}^{n-1} w^j}$ and expects to win on Battlefield 2 with probability $1-\mu_E^d\left(T_1^e \cup ... \cup T_{i-1}^e\right) = \frac{\sum_{j=i-1}^{n-i} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore his total expected payoff is greater than $\frac{\sum_{j=0}^{n} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, another contradiction. Therefore, $\mu_E^d(D_i^e)$ must equal zero.

As the above analysis holds for all $i = 1, 2, \ldots, n-1$, simple induction shows that the mass over all such T_i^e and D_i^e must equal $\frac{w^{i-1}}{\sum\limits_{j=0}^{n-1} w^j}$ and 0, respectively.

---

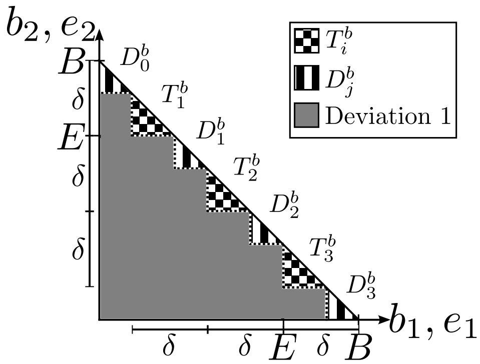
[46] As Blotto never plays in $b_1 \in \left(f_b^{-1}\left(g^{i-2}\left(E_2\right)\right), h^{n-i}(E_1)\right)$ and wins ties on Battlefield 1 when he plays $b_1 = \left(h^{n-i}\left(E_1\right)\right)$.

[47] As Blotto plays $\left(h^{n-i-1}(E_1), f\left(h^{n-i-1}(E_1)\right)\right)$with probability $\frac{w^{n-i-1}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$ when playing μ_B^* and $f\left(h^{n-i-1}(E_1)\right) = g\left(h^{n-i}\left(E_1\right)\right)$ by definition.

[48] Which he does with probability $\frac{w^{n-i}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$ in strategy μ_B^*

[49] Which he does with probability $\frac{w^{n-i}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$ in strategy $\mu_B^*(.)$

Figure B.11: Blotto Deviations in Region 3



The remaining mass of $\frac{w^{n-1}}{\sum_{j=0}^{n-1} w^j}$ must then be distributed over the only region left, T_n^e. Therefore, μ_E^d satisfies property 1e. We've already discussed why it must also satisfy $2e$. Therefore, $\mu_E^d \in \Omega^E$, and there can be no enemy strategies which are a part of some Nash equilibrium which do not satisfy our characterization.

We've ruled out any potential Enemy strategies that deviate from our characterization of possible Nash Equilibrium Enemy strategies. As the logic is quite similar, we rule out any potential Blotto deviations in our proof of Proposition 10 in a separate Appendix (Appendix Appendix B.) Therefore, the characterization is complete. Therefore, the complete set of Nash equilibria to any constant sum two battlefield Blotto game is the set of all pairs of Blotto and Enemy strategies satisfying our characterization. □

Appendix B. Proof of Completeness of the Blotto Strategies

The proof that we've characterized the complete set of Blotto's equilibrium strategies in the constant sum game proceeds much the same way as the proof of completeness for Enemy's strategies. However, due to the open boundaries of the T_i^e's, the proof is slightly more complicated.

Proposition 10. *In the constant sum game* ($w = w_B = w_E$) *any Blotto strategy which is a part of some Nash Equilibrium satisfies properties 1b and 2b.*

Proof. Suppose not. Then there exists at least one Blotto strategy which is a part of some Nash Equilibrium that does not satisfy properties 1b and 2b. Call such a strategy μ_B^d We've already shown how a strategy that satisfied property 1b, but violated property 2b would give Enemy an allocation offering a payoff higher than his constant equilibrium payoff. So, any uncharacterized Blotto strategy which is part of some Nash equilibrium must violate property 1b. In general, there are two ways Blotto could violate this property: He could sometimes play outside of T^b, or his μ_B could allocate play inappropriately within T^b. We break the former down into two separate deviations. The first type of deviations we consider **(Deviation 1)** are where Blotto mixes over allocations that send weakly less to both battlefields than some non-deviating allocation. Formally a strategy, $\mu_B^{d_1}$, exhibits **Deviation 1** if the following conditions hold for some S :

$$\mu_B^{d_1}(S) > 0 \tag{B.1}$$

$$S \cap T^b = \emptyset \tag{B.2}$$

$$\forall (b_1^*, b_2^*) \in S, \quad \exists (b_1, b_2) \in T^b \quad s.t. \quad b_1^* \leq b_1 \quad and \quad b_2^* \leq b_2 \tag{B.3}$$

The second condition implies that one of the two inequalities in the third holds strictly.

The next type of deviation we consider **(Deviation 2)** are the remaining feasible allocations outside of T^b. Specifically, these are the allocations that are outside T^b, and send strictly more to one battlefield than any allocation inside T^b. Formally a strategy, $\mu_B^{d_2}$, exhibits **Deviation 2** if the following conditions hold for some S:

$$\mu_B^{d_2}(S) > 0 \tag{B.4}$$

$$S \cap T^b = \emptyset \tag{B.5}$$

$$\forall (b_1^*, b_2^*) \in S, (b_1, b_2) \in T^b \quad either \quad b_1^* > b_1 \quad or \quad b_2^* > b_2 \tag{B.6}$$

Consider the following triangles:

$$D_i^b \equiv \{(b_1, b_2) : \ (b_1 > f^{-1}\left(p^{i-1}\left(E_2\right)\right)), (b_2 > f\left(h^{n-i-1}\left(E_1\right)\right)), (b_2 \leq f\left(b_1\right))\} \ \forall i = 0, 1, ..., n. \tag{B.7}$$

[50] A strategy, $\mu_B^{d_2}$, satisfying **Deviation 2** must allocate some mass over at least one of the D_j^b's as these are the only regions where conditions B.4-B.6 hold.

The last type of deviation we consider **(Deviation 3)** is simply where Blotto plays inappropriate mass over one of his T_i^b. Formally, a strategy $\mu_B^{d_3}$ exhibits **Deviation 3** if the following condition holds for at least one of Blotto's T_i^b's:

$$\mu_B^{d_3}\left(T_i^b\right) \neq \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}.$$

Because of Lemma 8 (Equilibrium Interchangeability), any Blotto strategy from any Nash equilibrium must form a Nash equilibrium with any Enemy strategy from Ω^E. Specifically, we consider the following sequence of Enemy strategies: For any $k = 1, 2, ...$ let μ_E^k be the strategy where in each T_i^e Enemy plays points $\left(f^{-1}\left(p^{i-2}\left(E_2\right)\right) + \frac{\epsilon}{k}, g\left(f^{-1}\left(p^{i-2}\left(E_2\right)\right) + \frac{\epsilon}{k}\right)\right)$ and $\left(h^{n-i}(E_1) - \frac{\epsilon}{k}, g\left(h^{n-i}(E_1) - \frac{\epsilon}{k}\right)\right)$ with probability $\frac{w^{i-1}}{2 \cdot \sum\limits_{j=0}^{n-1} w^j}$. Clearly this implies Enemy plays nowhere else.[51] Note

[50]Technically, for $j = 0$ the first inequality is weak, and for $j = n$ the second is weak.

[51]As $\sum_{i=1}^{n} 2 \cdot \frac{w_B^{i-1}}{2 \cdot \sum\limits_{j=0}^{n-1} w_B^j} = 1.$

that ϵ needs to be sufficiently small in order for μ_E^1 (and any other μ_E^k) to satisfy properties 1e and 2e. Assume that it is. Intuitively, this is a sequence of strategies that has Enemy playing arbitrarily close to the "endpoints" in his T_i^e's. In other words, for any T_i^e we will be able to find some μ_E^k where Enemy plays arbitrarily close to the intersection of his resource constraint and the either of other two bounds for T_i^e with strictly positive probability.

Now we are ready to start considering Blotto's potential deviations. **Deviation 1** has Blotto mix over allocations which send weakly less to both battlefields than than some allocation in some T_i^b. Since these deviating allocations are not themselves in T_i^b they must send strictly less to at least one battlefield. Consider a particular allocation (b_1^*, b_2^*) which satisfies **Deviation 1** relative to T_i^b. Suppose it sends strictly less to Battlefield 1, or $b_1^* < h^{n-i}(E_1)$ (and $b_2^* \leq f\left(h^{n-i}\left(E_1\right)\right)$). Let $\delta = h^{n-i}(E_1) - b_1^*$. There exists some k^* where $\delta > \frac{\epsilon}{k^*}$. Now, we know (b_1^*, b_2^*) cannot be a best response to $\mu_E^{k^*}$. Blotto could play $\left(h^{n-i}(E_1), f\left(h^{n-i}(E_1)\right)\right)$, but he plays (b_1^*, b_2^*) which strictly lowers his probability of winning on Battlefield 1 when Enemy plays $\mu_E^{k^*}$, and it does so without increasing Blotto's probability of winning on Battlefield 2. Therefore (b_1^*, b_2^*) provides a strictly lower payoff and cannot be a best response. Similar logic applies to (b_1^*, b_2^*) that send strictly less to Battlefield 2 (or where $b_1^* \leq f^{-1}\left(p^{i-1}\left(E_2\right)\right)$ and $b_2^* < p^{i-1}(E_2)$). Therefore, all allocations required for **Deviation 1** are not best responses to some μ_E^k. Therefore, no equilibrium Blotto strategy exhibits **Deviation 1**.

Intuitively **Deviation 1** represents Blotto placing some mass on allocations which, relative to some T_i^b, always send less to one battlefield, without increasing the allocation to the other. **Deviation 2** represents Blotto placing mass on allocations which, relative to any T_i^b, always send less to one battlefield, but increase the allocation to the other. **Deviation 3** has him playing an "incorrect" mass on some T_i^b.

Consider a deviating Blotto strategy μ_B^d that forms an Nash equilibrium with any $\mu_E \in \Omega^E$. First off realize that all allocations in D_0^b and D_n^b are not best responses to certain Nash equilibrium Enemy strategies. For instance, take an allocation $(b_1^*, b_2^*) \in D_0^b$. Blotto is increasing his Battlefield 2 allocation while reducing his Battlefield 1 allocation relative to T_1^b. However, in T_1^b Blotto was already guaranteeing victory on Battlefield 2 so this cannot be payoff improving. Clearly $b_1^* < h^{n-1}(E_1)$. We can find some $k^* \in \mathbb{N}$ such that $b_1^* < h^{n-1}(E_1) - \frac{\epsilon}{k^*} < h^{n-1}(E_1)$. As Enemy plays $h^{n-1}(E_1) - \frac{\epsilon}{k^*}$ on Battlefield 1 with positive probability in $\mu_E^{k^*}$, (b_1^*, b_2^*) must provide Blotto with a strictly lower payoff than $\left(h^{n-1}(E_1), f\left(h^{n-1}(E_1)\right)\right)$, which guarantees victory on Battlefield 2. Therefore, $\mu_B^d\left(D_0^b\right) = 0$. Similar logic implies that $\mu_B^d\left(D_n^b\right) = 0$.

We now simultaneously, inductively prove that neither **Deviations 2 or 3** are possible in a Nash equilibrium. Consider some $i \in \{1, 2, 3, ..., n-1\}$. Assume that

$$\forall j = 1, 2, \ldots, i-1, \quad \mu_B^d(T_j^b) = \mu_B^d(D_j^b) = 0 \; and \; \frac{w^{n-j}}{\sum\limits_{l=0}^{n-1} w^l} \tag{B.8}$$

In other words, there has not "yet" been a **Deviation 2** or **Deviation 3**.

Consider possible versions of **Deviation 3**. First suppose, $\mu_B^d\left(T_i^b\right) > \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$. Given equation B.8 and the fact that we've already ruled out **Deviation 1**, when Enemy plays in T_i^e his probability of winning on Battlefield 1 is $\mu_B^d\left(T_1^b \cup ... \cup T_{i-1}^b\right) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$. However, his probability of winning on Battlefield 2 is $1 - \mu_B^d(T_1^b \cup ... \cup T_i^b) < \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore, Enemy's total expected payoff is then strictly less than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$ which is his constant equilibrium payoff, a contradiction.

Second, suppose $\mu_B^d\left(T_i^b\right) < \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$. The only way Blotto plays a Battlefield 2 allocation weakly greater than $p^{i-1}(E_2)$ is if he plays in one of $T_1^b, ..., T_i^b$. Therefore the probability that Blotto plays a Battlefield 2 allocation strictly less than $p^{i-1}(E_2)$ is $1 - \mu_B^d(T_1^b \cup ... \cup T_i^b) > \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore, we can find some $\delta > 0$ such that the probability that Blotto plays a Battlefield 2 allocation strictly less than $p^{i-1}(E_2) - \delta$ is also greater than $\frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. For $k^* \in \mathbb{N}$ large enough $g\left(f^{-1}\left(p^{i-2}\left(E_2\right)\right) + \frac{\epsilon}{k^*}\right) > p^{i-1}(E_2) - \delta$. Enemy could therefore play $\left(f^{-1}\left(p^{i-2}\left(E_2\right)\right) + \frac{\epsilon}{k^*}, g\left(f^{-1}\left(p^{i-2}\left(E_2\right)\right) + \frac{\epsilon}{k^*}\right)\right) \in T_i^e$ and his probability of winning on Battlefield 1 would be $\mu_B(T_1^b \cup ... \cup T_{i-1}^b) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, while his probability of winning on Battlefield 2 would be greater than $\frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore his expected payoff would be greater than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, a contradiction. Therefore, $\mu_B^d\left(T_i^b\right) = \frac{w^{n-i}}{\sum\limits_{j=0}^{n-1} w^j}$.

Now consider a possible **Deviation 2**. Specifically, $\mu_B^d\left(D_i^b\right) > 0$. Note that all Battlefield 2 allocations in D_i^b are strictly greater than $f\left(h^{n-i-1}\left(E_1\right)\right)$. Define $S_i\left(\delta\right) = \left\{(b_1, b_2) : \ (b_1, b_2) \in D_i^b \ and \ b_2 \geq f\left(h^{n-i-1}\left(E_1\right)\right) + \delta\right\}$. We are then assured that $\exists \delta > 0$ sufficiently small that $\mu_B^d\left(S_i\left(\delta\right)\right) > 0$. Then we are also assured then that $\exists k^* \in \mathbb{N}$ such that $f\left(h^{n-i-1}\left(E_1\right)\right) + \delta > g\left(h^{n-i}(E_1) - \frac{\epsilon}{k^*}\right)$. Note that Enemy plays $\left(h^{n-i}(E_1) - \frac{\epsilon}{k}, g\left(h^{n-i}(E_1) - \frac{\epsilon}{k}\right)\right)$ in $\mu_E^{k^*}$. When he does so the probability that he wins on Battlefield 1 is $\mu_B(T_1^b \cup ... \cup T_{i-1}^b) = \frac{\sum_{j=n-(i-1)}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, but the probability he wins on Battlefield 2 is $1 - \mu_B(T_1^b \cup ... \cup T_{i-1}^b \cup S_i(\delta)) < \frac{\sum_{j=0}^{n-i-1} w^j}{\sum_{j=0}^{n-1} w^j}$. Therefore his expected payoff is strictly less than $\frac{\sum_{j=1}^{n-1} w^j}{\sum_{j=0}^{n-1} w^j}$, his constant equilibrium payoff, a contradiction. Therefore, μ_B^d does not violate Property 1b. Since we've already shown it can't violate property 2b, it is the case that $\mu_B^d \in \Omega^B$. □

Appendix C. Constant Payoffs and Equilibrium Interchangeability

Consider a two-player constant-sum game where Player 1 chooses a strategy $x \in X$ and Player 2 chooses a strategy $y \in Y$. Let $f^i(x, y)$ denote the expected payoffs to Player i when Player 1 plays x and Player 2 plays y. This game then satisfies the Constant payoffs property if in every equilibrium $\{x_j, y_j\}$, $f^i(x_j, y_j) = c_i$ for all $i = 1, 2$. Let Ω^i denote the set of player i strategies that are a part of some equilibrium. Lemma 8, Equilibrium Interchangeability, can then be stated as follows:

For all $x^* \in \Omega^1$ and all $y^* \in \Omega^2$, (x^*, y^*) constitutes a Nash Equilibrium.

Proof: Suppose not. Then $\exists x^* \in \Omega^1$, *and* $y^* \in \Omega^2$, such that (x^*, y^*) does not constitute an equilibrium. Therefore, at least one player must not be playing a best response. Without loss of generality assume it is player 1. There must exist an alternate $x' \in X$ such that $f^1(x', y^*) > f^1(x^*, y^*)$. As there exists an equilibrium where Player 2 plays y^* and Player 1 expects payoff c_1, $f^1(x', y^*) \leq c_1$. Therefore $f^1(x^*, y^*) < c_1$. As this game is constant-sum this means $f^2(x^*, y^*) > c_2$. This means Player 2 has a strategy available that when played against x^* provides a payoff greater than their constant equilibrium payoff. Therefore $x^* \notin \Omega^1$. This is a contradiction, and therefore Lemma 8 is true.